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Solving non-perturbative flow equations

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Abstract

Non- perturbative exact flow equations describe the scale dependence of the effective average action. We present a numerical solution for an approximate form of the flow equation for the potential in a three-dimensional N-component scalar field theory. The critical behaviour, with associated critical exponents, can be inferred with good accuracy.

Exact non-perturbative renormalization group equations describing the scale dependence of some type of effective action have been known for a long time [1]. They account for the consecutive inclusion of fluctuations in a field theory. There exist many versions of such equations which are, if exact, all equivalent, since they all describe in one way or another properties of the (Euclidean) functional integral which defines the theory. The most difficult part, however, is not so much to derive an exact flow equation - this usually follows from simple manipulations of the functional integral. The challenge is rather to find a formulation which can be used for practical computations beyond perturbation theory. For such practical purposes the most general form of the effective action has always to be truncated. Solutions of the truncated equations are only approximations to the exact flow equations. If a small parameter is available one can often organize a series of truncations as a power series in the small parameter - this is how usual perturbation theory is recovered. If no small parameter is known a priori one has to use as much as possible the knowledge of properties of the model in order to conceive a useful truncation. This is where it becomes important what effective action is selected and what the particular form of the flow equation is. Only if the effective action has a simple physical meaning so that its couplings can also be understood by other methods, and if the evolution equation has a form which incorporates directly the most prominent properties of the system, there is a realistic chance of getting a working non-perturbative tool. Beyond perturbation theory exact flow equations should, therefore, not be viewed as mainly a consistent (but unsolvable) mathematical system, but rather as a convenient starting point for appropriate non-perturbative expansion methods.

Typically, the truncated flow equations constitute a system of non-linear partial differential equations which can be solved by numerical methods. It is the aim of this letter to report on the development of algorithms which are adapted to the specific numerical problems of this approach. As an example we investigate here the N-component Heisenberg models in three dimensions. This is a classical testing ground for non-perturbative methods. We will extract the critical exponents of these models from a numerical solution of the flow equation for the scale dependent effective potential, as obtained from the lowest order in a derivative expansion of the effective action.

Recently the average action Γ_k [2] has been proposed as the free energy with an infrared cutoff $\sim k$. It is formulated in continuous space so that all symmetries of the model are preserved. More precisely the average action includes the effects of all fluctuations with momenta $q^2 > k^2$. In the limit $k \to 0$ it becomes the standard effective action (the generating functional of the 1PI Green functions), while for $k \to \infty$ it equals the classical action. There is a simple functional integral representation [2] of Γ_k also for $k > 0$ such that its couplings can, in principle, also be estimated by alternative methods. The average action is a coarse-grained free energy in the sense that short distance fluctuations are already integrated out. [1] The exact non-perturbative flow equation for Γ_k takes the

[1] One should not use Γ_k directly to replace the action in the functional integral, as this is the role of the "cutoff action" used in earlier approaches [1]. For this purpose an explicit ultraviolet cutoff term has to be added in order to prevent double counting of the short distance modes. This cutoff term is easily computed in simple theories of scalars and fermions [2], much more involved for an abelian gauge theory

simple form of a renormalization group improved one-loop equation [4]

$$\partial_t \Gamma_k = k\frac{\partial}{\partial k}\Gamma_k = \frac{1}{2}\mathrm{Tr}\tilde{\partial}_t \ln\left(\Gamma_k^{(2)} + R_k\right), \tag{1}$$

where $\tilde{\partial}_t$ acts only on the infrared cutoff piece R_k as

$$\tilde{\partial}_t \ln\left(\Gamma_k^{(2)} + R_k\right) = \left(\Gamma_k^{(2)} + R_k\right)^{-1} k\frac{\partial}{\partial k}R_k. \tag{2}$$

The trace involves a momentum integration and summation over internal indices. Most importantly, the relevant infrared properties appear directly in the form of the exact inverse average propagator $\Gamma_k^{(2)}$, which is the matrix of second functional derivatives with respect to the fields. There is always only one momentum integration - multi-loops are not needed - which is, for suitable R_k, both infrared and ultraviolet finite.

The flow equation (1) seems to be a good starting point for a non-perturbative approach. Nevertheless, it remains a complicated functional differential equation without any chance to be solved exactly. Approximate solutions need truncations - and the crux of the problem lies there. One possibility is to keep only a few invariants in Γ_k and, thereby, reduce eq. (1) to a finite set of ordinary differential equations for a finite number of couplings. Very satisfactory results have already been obtained this way [5]. Much more information is contained if, instead of a finite number of couplings, arbitrary functions of one or several parameters are considered. Examples are an arbitrary field dependence of the average potential $U_k(\rho)$ in scalar theories (with $\rho = \frac{1}{2}\phi^2$) [5, 6], or an arbitrary momentum dependence of the two-point function $G(p)$ [7] or even the four-point function $\lambda(q_1, q_2, q_3, q_4)$ [8]. The functional differential equation becomes then a partial differential equation for a function of at least two variables, e.g. $U(\rho, t) = U_k(\rho)$, or a system of partial differential equations for several such functions. If one does not want to resort to further approximations at this level, one needs appropriate tools for solving this type of partial differential equations. Analytical solutions can be found only in certain limiting cases and for most purposes numerical solutions seem the adequate tool. This is not so straightforward as it may seem at first sight, since the solutions of the differential equations have in general a highly unstable character due to the presence of relevant parameters. Fine tuning of initial conditions is needed in order to be near a phase transition - or, in particle physics language, to have renormalized masses much smaller than the ultraviolet cutoff. The algorithm for a numerical solution must be compatible with this situation and guarantee numerical stability of the critical solution which corresponds to the phase transition.

In this letter we demonstrate the capacities of such a non-perturbative method by computing the k-dependent average potential $U_k(\rho)$ for an N-component scalar field theory directly in three dimensions. For $k \to 0$ this gives the free energy which encodes the equation of state of the system. In the phase with spontaneous symmetry breaking the minimum of the potential occurs for $k = 0$ at $\rho_0 \neq 0$. The massless Goldstone excitations around this minimum are notoriously difficult to treat by alternative methods. For

[3], and not yet found for non-abelian gauge theories.

example, the standard loop expansion is order by order highly infrared divergent. In the symmetric phase the minimum of $U_k(\rho)$ ends at $\rho_0 = 0$ for $k = 0$. The two phases are separated by a scaling solution for which U_k/k^3 becomes independent of k once expressed in terms of a suitably rescaled field variable $\tilde{\rho}$.

Our truncation is the lowest order in a systematic derivative expansion of Γ_k [2, 5, 6]

$$\Gamma_k = \int d^d x \Big\{ U_k(\rho) + \frac{1}{2} Z_k \partial^\mu \phi_a \partial_\mu \phi^a \Big\}. \tag{3}$$

Here ϕ^a denotes the N-component real scalar field and $\rho = \frac{1}{2}\phi^a\phi_a$. We keep for the potential term the most general $O(N)$-symmetric form $U_k(\rho)$, whereas the wave function renormalization is approximated by one k-dependent parameter. Next order in the derivative expansion would be the generalization to a ρ-dependent wavefunction renormalization $Z_k(\rho)$ plus a function $Y_k(\rho)$ accounting for a possible different index structure of the kinetic term for $N \geq 2$ [2, 5]. Going further would require the consideration of terms with four derivatives and so on. For the three-dimensional scalar theory the anomalous dimension η is known to be small and the derivative expansion is, therefore, expected to give a reliable approximation [5]. The main reason is that for $\eta = 0$ the kinetic term in the k-dependent inverse propagator must be exactly proportional to q^2 both for $q^2 \to 0$ and $q^2 \to \infty$. This holds for arbitrary constant "background" field ϕ^a. Similar, although less stringent, arguments concern the smallness of the ρ-dependence of the kinetic term [5]. For the scaling solution for $N = 1$ this weak ρ-dependence has been established explicitly [6]. We finally mention that η is proportional to a small parameter $\lambda/8\pi^2$, where λ is a suitably defined quartic scalar coupling. (For the scaling solution λ takes a fixed point value $\lambda_*/8\pi^2 = 0.12$ for $N = 1$.) We expect that the derivative expansion can be understood as an expansion in this small parameter. However, since some of the other parameters effectively behave $\sim \lambda^{-1}$ this expansion is not equivalent to the usual perturbative polynomial series in λ.

For a study of the behaviour in the vicinity of the phase transition it is convenient to work with dimensionless renormalized fields [2]

$$\begin{aligned} \tilde{\rho} =& Z_k k^{2-d} \rho \\ u_k(\tilde{\rho}) =& k^{-d} U_k(\rho). \end{aligned} \tag{4}$$

With the truncation of eq. (3) the exact evolution equation for $u_k' \equiv \partial u_k/\partial\tilde{\rho}$ [5] reduces then to the partial differential equation

$$\begin{aligned} \frac{\partial u_k'}{\partial t} =& \ (-2+\eta) u_k' + (d-2+\eta)\tilde{\rho} u_k'' \\ & - 2v_d(N-1) u_k'' l_1^d(u_k';\eta) - 2v_d(3u_k'' + 2\tilde{\rho} u_k''') l_1^d(u_k' + 2\tilde{\rho} u_k'';\eta), \end{aligned} \tag{5}$$

where $t = \ln(k/\Lambda)$, with Λ the ultraviolet cutoff of the theory. The anomalous dimension η is defined by

$$\eta = -\frac{\partial}{\partial t} \ln Z_k \tag{6}$$

[2]We keep the number of dimensions d arbitrary and specialize only later to $d = 3$.

and

$$v_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right), \tag{7}$$

with $v_3 = 1/8\pi^2$. The "threshold" functions $l_n^d(w;\eta)$ result from the momentum integration on the r.h.s. of eq. (1), and read for $n \geq 1$, with $y = q^2/k^2$

$$\begin{aligned} l_n^d(w;\eta) = & -n\int_0^\infty dy y^{\frac{d}{2}+1}\frac{\partial r(y)}{\partial y}\left[y(1+r(y))+w\right]^{-(n+1)} \\ & -\frac{n}{2}\eta\int_0^\infty dy y^{\frac{d}{2}} r(y)\left[y(1+r(y))+w\right]^{-(n+1)}. \end{aligned} \tag{8}$$

Here $r(y)$ depends on the choice of the momentum dependence of the infrared cutoff and we employ

$$r(y) = \frac{e^{-y}}{1-e^{-y}}. \tag{9}$$

This choice has the property $\lim_{q^2\to 0} R_k/Z_k k^2 = \lim_{y\to 0} y r(y) = 1$, whereas for $q^2 \gg k^2$ the effect of the infrared cutoff is exponentially suppressed. The "threshold" functions account for the decoupling of modes with mass larger than k and decrease rapidly for $w \gg 1$. For our purpose we use numerical fits for these functions. Finally, the anomalous dimension is given in our truncation by

$$\eta(k) = \frac{16 v_d}{d}\kappa\lambda^2 m_{2,2}^d(2\lambda\kappa), \tag{10}$$

with κ the location of the minimum of the potential and λ the quartic coupling

$$\begin{aligned} u_k'(\kappa) =& 0 \\ u_k''(\kappa) =& \lambda. \end{aligned} \tag{11}$$

The function $m_{2,2}^d$ is given by [5]

$$\begin{aligned} m_{2,2}^d(w) = & \int_0^\infty dy y^{\frac{d}{2}-2}\frac{1+r+y\frac{\partial r}{\partial y}}{(1+r)^2\left[(1+r)y+w\right]^2} \\ & \left\{2y\frac{\partial r}{\partial y} + 2\left(y\frac{\partial}{\partial y}\right)^2 r - 2y^2\left(1+r+y\frac{\partial r}{\partial y}\right)\frac{\partial r}{\partial y}\left[\frac{1}{(1+r)y}+\frac{1}{(1+r)y+w}\right]\right\}. \end{aligned} \tag{12}$$

We point out that the argument $2\lambda\kappa$ turns out generically to be of order one for the scaling solution. Therefore, $\kappa \sim \lambda^{-1}$ and the mass effects are important, in contrast to perturbation theory where they are treated as small quantities $\sim \lambda$.

Our aim is the development of algorithms for the numerical solution of the partial differential equation (5), and a comparison with previously used expansion methods. Near the phase transition the trajectory spends most of the "time" t in the vicinity of the critical

k-independent scaling solution given by $\partial_t u'_*(\tilde{\rho}) = 0$. [3] Only at the end of the running the "near-critical" trajectories deviate from the scaling solution. For $k \to 0$ they either end up in the symmetric phase with $\kappa = 0$ and positive constant mass term m^2 such that $u'_k(0) \sim m^2/k^2$; or they lead to a non-vanishing constant ρ_0 indicating spontaneous symmetry breaking with $\kappa \to Z_0 k^{2-d}\rho_0$. The equation of state involves the potential $U_0(\rho)$ for temperatures away from the critical temperature. Its computation requires the solution for the running away from the critical trajectory which involves the full partial differential equation (5). We have developed two alternative numerical approaches which we briefly describe in the following. For both methods we replace the variable $\tilde{\rho}$ by a discrete set of points $\tilde{\rho}_i$, $i = 1, ..., m$.

I) Consider first what happens if we make a Taylor expansion of u_k around some arbitrary point $\tilde{\rho}_i$

$$u_k(\tilde{\rho}) = \sum_{n=0}^{\infty} \frac{1}{n!} u_i^{(n)} (\tilde{\rho} - \tilde{\rho}_i)^n, \tag{13}$$

with $u_i^{(n)}(k) = u_k^{(n)}(\tilde{\rho}_i)$. The potential is then described by infinitely many couplings $u_i^{(n)}(k)$. The flow equations for these couplings are obtained from appropriate $\tilde{\rho}$-derivatives of eq. (5) evaluated at $\tilde{\rho} = \tilde{\rho}_i$. We observe that the flow equation for $u_i^{(1)}$ involves $u_i^{(1)}$, $u_i^{(2)}$ and $u_i^{(3)}$, the one for $u_i^{(2)}$ needs in addition $u_i^{(4)}$, and the system is never closed. Our first approach considers at every point $\tilde{\rho}_i$ the differential equations for $u_i^{(1)}$ and $u_i^{(2)}$. The "missing" couplings $u_i^{(3)}$ and $u_i^{(4)}$ appearing in these equations are determined by matching the expansion around $\tilde{\rho}_i$ with similar expansions around different points $\tilde{\rho}_{j\neq i}$. More precisely, the matching is done by equating fourth order polynomial expansions of $u'_k(\tilde{\rho})$ around two neighbouring points $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}$ at half-distance, and similarly for $u''_k(\tilde{\rho})$

$$\begin{aligned}
&(u'_k)_i\left(\frac{\tilde{\rho}_i + \tilde{\rho}_{i+1}}{2}\right) \equiv u_i^{(1)} + u_i^{(2)}\frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{2} + u_i^{(3)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^2}{8} + u_i^{(4)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^3}{48} \\
= \;& (u'_k)_{i+1}\left(\frac{\tilde{\rho}_i + \tilde{\rho}_{i+1}}{2}\right) \equiv u_{i+1}^{(1)} - u_{i+1}^{(2)}\frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{2} + u_{i+1}^{(3)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^2}{8} - u_{i+1}^{(4)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^3}{48}
\end{aligned} \tag{14}$$

$$\begin{aligned}
&(u''_k)_i\left(\frac{\tilde{\rho}_i + \tilde{\rho}_{i+1}}{2}\right) \equiv u_i^{(2)} + u_i^{(3)}\frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{2} + u_i^{(4)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^2}{8} \\
= \;& (u''_k)_{i+1}\left(\frac{\tilde{\rho}_i + \tilde{\rho}_{i+1}}{2}\right) \equiv u_{i+1}^{(2)} - u_{i+1}^{(3)}\frac{\tilde{\rho}_{i+1} - \tilde{\rho}_i}{2} + u_{i+1}^{(4)}\frac{(\tilde{\rho}_{i+1} - \tilde{\rho}_i)^2}{8}.
\end{aligned} \tag{15}$$

Combining eqs. (14) and (15) at all $m-1$ intermediate points ($i = 1, ..., m$) gives an algebraic system of $2m-2$ equations for the $2m$ unknowns $u_i^{(3)}$ and $u_i^{(4)}$. In order to obtain the remaining two necessary equations, we also match for the initial and end points the third derivative of the expansions, at $(\tilde{\rho}_1+\tilde{\rho}_2)/2$ and $(\tilde{\rho}_{m-1}+\tilde{\rho}_m)/2$ respectively. Together these equations make up an algebraic system which has a unique solution. This

[3] The resulting ordinary differential equation for $u_*(\tilde{\rho})$ has already been solved numerically for a somewhat different choice of the infrared cutoff [6].

allows one to express the couplings $u_i^{(3)}$ and $u_i^{(4)}$ as functions of the couplings $u_j^{(1)}$ and $u_j^{(2)}$, $j = 1, ..., m$. The integration of the remaining system of $2m$ differential equations for the couplings $u_i^{(1)}$, $u_i^{(2)}$ is done with the fifth order Runge-Kutta algorithm using the embedded fourth order method for precision control. The function $u_k'(\tilde{\rho})$ is finally reconstructed by patching the fourth order Taylor expansions around $\tilde{\rho}_i$ together at half-distance around neighbouring points. The polynomial patching improves with decreasing distance between neighbouring expansion points. This can be used to check the stability of numerical results.

We note that the matching conditions of eqs. (14), (15) guarantee continuity of $u_k'(\tilde{\rho})$ and $u_k''(\tilde{\rho})$. (Further smoothening could be applied for $u_k^{(3)}(\tilde{\rho})$ if needed.) Furthermore, they imply that the differential equations for $u_i^{(1)}$ and $u_i^{(2)}$ do not only incorporate information from directly neighbouring points, but from the whole range of points, as implied by the algebraic solution for $u_i^{(3)}$ and $u_i^{(4)}$. Nevertheless, one observes that the contributions of $u_j^{(1,2)}$ to $u_i^{(3,4)}$ $(j \neq i)$ rapidly decrease with increasing $|i - j|$. The decoupling from distant points could be used to obtain an approximate expression for $u_i^{(3,4)}$ which becomes useful if a large number of expansion points is considered. For the computation of the critical exponents we expand here the potential around 10 points. As long as the minimum of the potential is located away from the origin we choose expansion points $\tilde{\rho}_i$ proportional to κ. If the minimum is at or very close to the origin we use instead expansion points corresponding to fixed values of ρ instead of fixed $\tilde{\rho}$.

II) For the alternative approach we first consider a standard discretized version of eq. (5)

$$k\frac{u_i^{(1)}(k+\Delta k) - u_i^{(1)}(k)}{\Delta k} = F\left[u_i^{(1)}(k), u_i^{(2)}(k), u_i^{(3)}(k)\right]. \tag{16}$$

The higher $\tilde{\rho}$-derivatives $u_i^{(2)}(k), u_i^{(3)}(k)$ can be inferred from the differences of the values of $u_{i\pm1}^{(1)}(k)$ and the integration seems straightforward. The main problem with this approach is the appearance of numerical instabilities, with the numerical solution becoming strongly oscillating after a few integration steps. These instabilites are suppressed if the r.h.s. of eq. (16) is evaluated at $k+\Delta k$ instead of k. This leads us to replace the partial differential equation (5) by the system of m algebraic equations

$$k\frac{u_i^{(1)}(k+\Delta k) - u_i^{(1)}(k)}{\Delta k} = F\left[u_i^{(1)}(k+\Delta k), u_i^{(2)}(k+\Delta k), u_i^{(3)}(k+\Delta k)\right], \tag{17}$$

with the higher $\tilde{\rho}$-derivatives expressed in terms of $u_{i\pm1}^{(1)}(k+\Delta k)$. [4] At every step the m unknowns $u_i^{(1)}(k+\Delta k)$ can be calculated with the Newton-Raphson method. Further improvement in the accuracy can be achieved by making use of the solution at k and $k-\Delta k$ for the calculation of the solution at $k+\Delta k$ according to

$$k\frac{3u_i^{(1)}(k+\Delta k) - 4u_i^{(1)}(k) + u_i^{(1)}(k-\Delta k)}{2\Delta k} = F\left[u_i^{(1)}(k+\Delta k), u_i^{(2)}(k+\Delta k), u_i^{(3)}(k+\Delta k)\right].$$

[4] For the end points $\tilde{\rho}_1$, $\tilde{\rho}_m$, the higher derivatives can be obtained at the same level of accuracy in terms of $u_{1,2,3,4}^{(1)}$ and $u_{m,m-1,m-2,m-3}^{(1)}$ respectively.

(18)

The expression on the l.h.s. approximates the k-derivative at $k + \Delta k$ with an accuracy of $O(|\Delta k|^2)$. We use 60 points for the discretization of the variable $\tilde{\rho}$ and a varying number of k steps (around 500) until stability of the results is obtained.

The use of two algorithms for the integration of eq. (5) provides a good check for possible systematic numerical uncertainties. The two methods give results which agree at the 0.3 % level and the difference is most likely due to the different use of fits for the "threshold" functions. We expect the numerical solution to be an approximation of the solution of the partial differential equation (5) with the same level of accuracy. This has to be compared with the uncertainty induced by the omission of the higher derivative terms in the average action. The latter is expected to be of the order of η, as we discussed earlier, and is the main source of error for the results presented in the following.

In fig. 1 we present the results of the numerical integration of eq. (5) for $d = 3$ and $N = 1$. The function $u'_k(\tilde{\rho})$ is plotted for various values of $t = \ln(k/\Lambda)$. The evolution starts at $k = \Lambda$ ($t = 0$) where the average potential is equal to the classical potential (no effective integration of modes has been performed). We start with a quartic classical potential parametrized as

$$u'_\Lambda(\tilde{\rho}) = \lambda_\Lambda(\tilde{\rho} - \kappa_\Lambda). \tag{19}$$

We arbitrarily choose $\lambda_\Lambda = 0.1$ and fine tune κ_Λ so that a scaling solution is approached at later stages of the evolution. There is a critical value $\kappa_{cr} \simeq 6.396 \times 10^{-2}$ for which the evolution leads to the scaling solution without ever deviating from it. For the results in fig. 1 a value κ_Λ slightly smaller than κ_{cr} is used. As k is lowered (and t turns negative), $u'_k(\tilde{\rho})$ deviates from its initial linear shape. Subsequently it evolves towards a form which is independent of k and corresponds to the scaling solution $\partial_t u'_*(\tilde{\rho}) = 0$. It spends a long "time" t - which can be rendered arbitrarily long through appropriate fine tuning of κ_Λ - in the vicinity of the scaling solution. During this "time", the minimum of the potential $u'_k(\tilde{\rho})$ takes a fixed value κ_*, while the minimum of $U_k(\rho)$ evolves towards zero according to

$$\rho_0(k) = k\kappa_*/Z_k. \tag{20}$$

The longer $u'_k(\tilde{\rho})$ stays near the scaling solution, the smaller the resulting value of $\rho_0(k)$ when the system deviates from it. As this value determines the mass scale for the renormalized theory at $k = 0$, the scaling solution governs the behaviour of the system very close to the phase transition, where the characteristic mass scale goes to zero. Another important property of the "near-critical" trajectories, which spend a long "time" t near the scaling solution, is that they become insensitive to the details of the classical theory which determine the initial conditions for the evolution. After $u'_k(\tilde{\rho})$ has evolved away from its scaling form $u'_*(\tilde{\rho})$, its shape is independent of the choice of λ_Λ for the classical theory. This property gives rise to the universal critical behaviour near second order phase transitions. For the solution depicted in fig. 1 $u_k(\tilde{\rho})$ evolves in such a way that its minimum runs to zero with $u'_k(0)$ subsequently increasing. Eventually the theory settles down in the symmetric phase with a positive constant renormalized mass term $m^2 = k^2 u'_k(0)$

as $k \to 0$. Another possibility is that the system ends up in the phase with spontaneous symmetry breaking. In this case κ grows in such a way that $\rho_0(k)$ approaches a constant value for $k \to 0$.

The approach to the scaling solution and the deviation from it can also be seen in fig. 2. The evolution of the running parameters $\kappa(t)$, $\lambda(t)$ starts with their initial classical values, leads to fixed point values κ_*, λ_* near the scaling solution, and finally ends up in the symmetric phase (κ runs to zero). Similarly the anomalous dimension $\eta(k)$, which is given by eq. (10), takes a fixed point value η_* when the scaling solution is approached. During this part of the evolution the wave function renormalization is given by

$$Z_k \sim k^{-\eta_*} \tag{21}$$

according to eq. (6). When the parts of the evolution towards and away from the fixed point become negligible compared to the evolution near the fixed point - that is, very close to the phase transition - eq. (21) becomes a very good approximation for sufficiently low k. This indicates that η_* can be identified with the critical exponent η. For the solution of fig. 2 ($N = 1$) we find $\kappa_* = 4.07 \times 10^{-2}$, $\lambda_* = 9.04$ and $\eta_* = 4.4 \times 10^{-2}$.

As we have already mentioned the details of the renormalized theory in the vicinity of the phase transition are independent of the classical coupling λ_Λ. Moreover, the critical theory can be parametrized in terms of critical exponents [9], an example of which is the anomalous dimension η. These exponents are universal quantities which depend only on the dimensionality of the system and its internal symmetries. For our three-dimensional theory they depend only on the value of N and can be easily extracted from our results. We concentrate on the exponent ν, which parametrizes the behaviour of the renormalized mass in the critical region. The other exponents are not independent quantities, but can be determined from η and ν through universal scaling laws [9]. We define the exponent ν through the renormalized mass term in the symmetric phase

$$m^2 = \frac{1}{Z_k}\frac{dU_k(0)}{d\rho} = k^2 u_k'(0) \qquad \text{for} \quad k \to 0. \tag{22}$$

The behaviour of m^2 in the critical region depends only on the distance from the phase transition, which can be expressed in terms of the difference of κ_Λ from the critical value κ_{cr} for which the renormalized theory has exactly $m^2 = 0$. The exponent ν is determined from the relation

$$m^2 \sim |\delta\kappa_\Lambda|^{2\nu} = |\kappa_\Lambda - \kappa_{cr}|^{2\nu}. \tag{23}$$

For a determination of ν from our results we calculate m^2 for various values of κ_Λ near κ_{cr}. We subsequently plot $\ln(m^2)$ as a function of $\ln|\delta\kappa_\Lambda|$. This curve becomes linear for $\delta\kappa_\Lambda \to 0$ and we obtain ν from the constant slope. In the past the critical exponents of the $O(N)$-symmetric theory were calculated from truncated versions of the partial differential equation (5) [5]. The strategy was to turn eq. (5) into an infinite system of ordinary differential equations for the coefficients of a Taylor expansion analogous to eq. (13) around the "running" minimum of the potential. This infinite system was approximately solved by neglecting $\tilde{\rho}$-derivatives of $u_k(\tilde{\rho})$ higher than a given order. The apparent

convergence of the procedure was checked by enlarging the level of truncation. We now have an alternative way of estimating the accuracy of this method. Our numerical solution of the partial differential equation (5) corresponds to an infinite level of truncation where all the higher derivatives are taken into account. In table 1 we present results obtained through the procedure of successive truncations and through our numerical solution for $N = 3$. We give the values of κ, λ, $u_k^{(3)}(\kappa)$ for the scaling solution and the critical exponents η, ν. We observe how the results stabilize as more $\tilde{\rho}$-derivatives of $u_k(\tilde{\rho})$ at $\tilde{\rho} = \kappa$ and the anomalous dimension are taken into account. The last line gives the results of our numerical solution of eq. (5). By comparing with the previous line we conclude that the inclusion of all the $\tilde{\rho}$-derivatives higher than $u_k^{(6)}(\kappa)$ and the term $\sim \eta$ in the "threshold" function of eq. (8) generates an improvement of less than 1 % for the results. This is a lot smaller than the error induced by the omission of the higher derivative terms in the average action, which typically generates an uncertainty of the order of the anomalous dimension. In table 2 we compare our values for the critical exponents with more accurate results obtained with other methods (such as the ϵ-expansion, summed perturbation theory at fixed dimension, lattice calculations and the $1/N$-expansion). As expected η is rather poorly determined since it is the quantity most seriously affected by the omission of the higher derivative terms in the average action. The exponent ν is in agreement with the known results at the 1-5 % level, with a discrepancy roughly equal to the value of η for various N.

In conclusion, the shape of the average potential is under good quantitative control for every scale k. This permits a quantitative understanding of the most important properties of the system at every length scale. Our investigation is not restricted to the behaviour near the phase transition on which we have concentrated here because it is the most difficult to handle numerically. Also the initial form of the potential does not have to be of the quartic form of eq. (19). Arbitrary general short-distance potential can be studied. For example, the tricritical point for the transition to a first order phase transition can be investigated with our numerical methods. Furthermore, the flow equation is well defined for arbitrary continuous d. This will permit an explicit check of the validity of the ϵ-expansion for more detailed quantities characterizing the equation of state.

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Tables

	κ_*	λ_*	$u_*^{(3)}$	η	ν
a	6.57×10^{-2}	11.5			0.745
b	8.01×10^{-2}	7.27	52.8		0.794
c	7.86×10^{-2}	6.64	42.0	3.6×10^{-2}	0.760
d	7.75×10^{-2}	6.94	43.5	3.8×10^{-2}	0.753
e	7.71×10^{-2}	7.03	43.4	3.8×10^{-2}	0.752
f	7.64×10^{-2}	7.07	44.2	3.8×10^{-2}	0.747

Table 1: The minimum κ of the potential $u_k(\tilde{\rho})$, the derivatives $\lambda = u'(\kappa)$, $u_k^{(3)}(\kappa)$ for the scaling solution, and the critical exponents η and ν, in various approximations: (a)-(e) from ref. [5] and (f) from the present letter. $N = 3$.
a) Truncation where only the evolution of κ and λ is considered and higher derivatives of the potential and the anomalous dimension are neglected.
b) κ, λ, $u_k^{(3)}(\kappa)$ are included.
c) κ, λ, $u_k^{(3)}(\kappa)$ are included and η is approximated by eq. (10).
d) with five parameters: κ, λ, $u_k^{(3)}(\kappa)$, $u_k^{(4)}(\kappa)$ and η.
e) as in d) and in addition $u_k^{(5)}(\kappa)$, $u_k^{(6)}(\kappa)$ are estimated.
f) The partial differential equation (5) for $u_k'(\tilde{\rho})$ is solved numerically and η is approximated by eq. (10).

N	ν		η	
1	0.643	$0.6300(15)^a$ $0.6310(15)^b$ $0.6305(15)^c$	0.044	$0.032(3)^a$ $0.0375(25)^b$
2	0.697	$0.6695(20)^a$ $0.671(5)^b$ $0.672(7)^c$	0.042	$0.033(4)^a$ $0.040(3)^b$
3	0.747	$0.705(3)^a$ $0.710(7)^b$ $0.715(20)^c$	0.038	$0.033(4)^a$ $0.040(3)^b$
4	0.787		0.034	
10	0.904	0.877^d	0.019	0.025^d
100	0.990	0.989^d	0.002	0.003^d

Table 2: Critical exponents ν and η for various values of N. For comparison we list results obtained with other methods as summarized in [10] and [11]:
a) From summed perturbation series in fixed dimension 3 at six-loop order.
b) From the ϵ-expansion at order ϵ^5.
c) From lattice calculations.
d) From the $1/N$-expansion at order $1/N^2$.

Figures

Fig. 1 The evolution of $u_k'(\tilde{\rho})$ as k is lowered from Λ to zero. The initial conditions (bare couplings) have been chosen such that the scaling solution is approached before the system evolves towards the symmetric phase with $u_k'(0) > 0$. $N = 1$.

Fig. 2 The evolution of κ, λ and η for the solution of fig. 1.

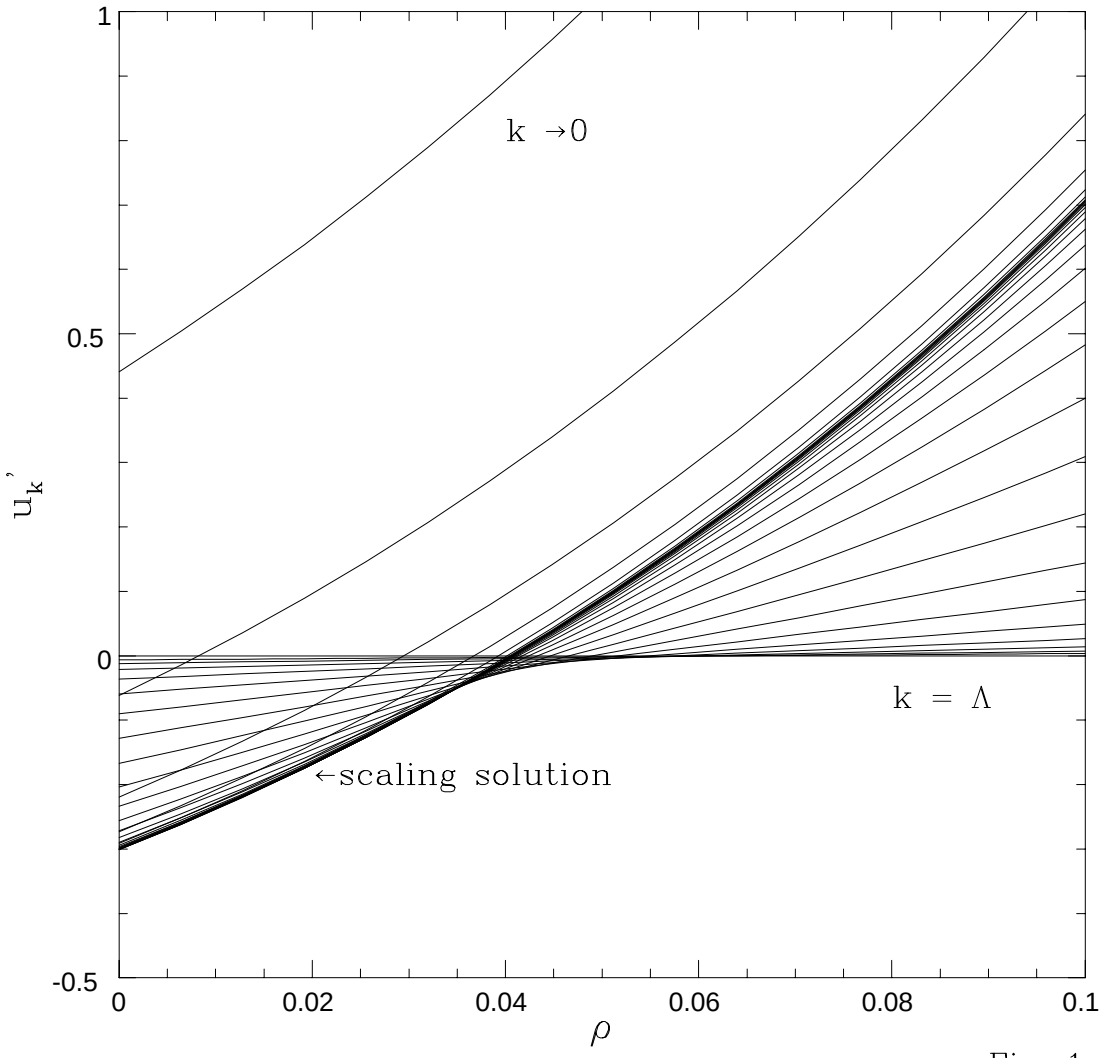

Fig. 1

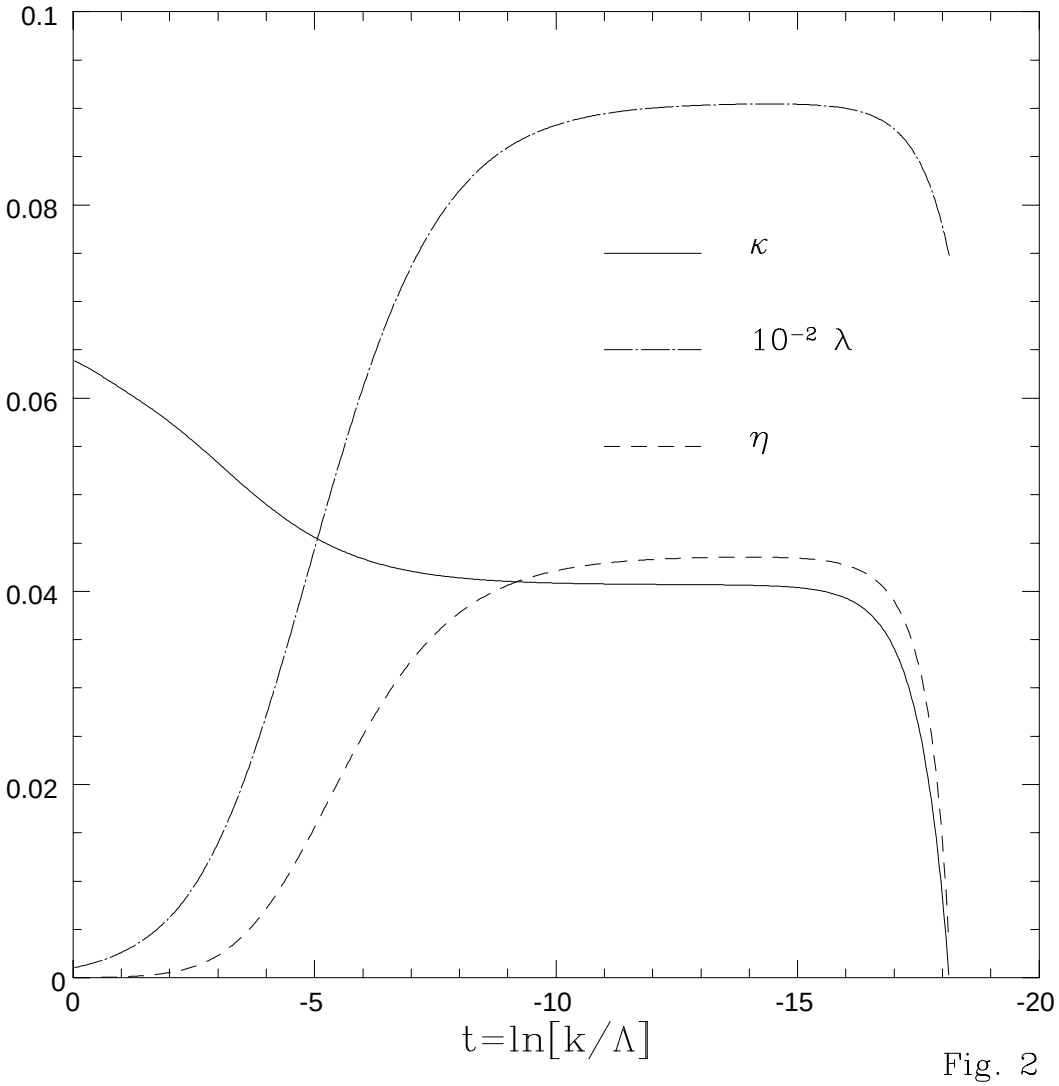

Fig. 2